Equivalariance on Discrete Space and Yang-Mills-Higgs Model

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Abstract

We introduce the basic equivariant quantity Q in the gauge theory on the noncommutative descrete Z_2 space, which plays an important role for the equivariant dimensional reduction. If the gauge configuration of the ground state on the extra dimensional space is described by the equivariant Q, then the extra dimensional space is invisible. Especially, using the equivariance principle, we show that the Yang-Mills theory on $R^2 \times Z_2$ space is equivalent to the Yang-Mills-Higgs model on R^2 space. It can be said that this model is the simplest model of this type.

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1 Introduction

Discovery of Higgs boson has brought about a lot of activities concerned with the origin of this particle. Although various models have been proposed, the convincing one does not seem to exist. Higgs boson was introduced as a particular particle that spontaneously violates the gauge symmetry. Thus, it is important to investigate the origin of this boson. One candidate could be to consider the extra dimensions to our real space that we are recognizing and to reduce the extra dimensions by considering the equivariance of symmetries involved[1, 2, 3, 4, 5].

Equivariance implies that the symmetry of the real space is related to that of the internal space, so that the shifted point in the real space can be sent back to the original point by the symmetry transformation in the internal space. Thus the extra dimensional space, even when it is there, could have been unobserved if the symmetry of the extra space is equivariant to the gauge symmetry of the real space, that we are actually living in. The gauge fields in the extra dimensional space are observed as the Higgs fields in the real space. In other words, the Higgs fields can be considered as the gauge fields in the extra dimensional space.

Consider, for example, the case of $R^2 \times S^2$, where the real space is R^2 and the extra space is S^2[1, 2]. Equivariance on S^2 implies that pure Yang-Mills (YM) theory is recognized as Yang-Mills-Higgs (YMH) model on R^2. Thus, self dual (SD) equation on $R^2 \times S^2$ is equivalent to Bogomol'nyi-Prasad-Sommerfield (BPS) equation on R^2[6]. Invisibility of the extra dimensions is guaranteed by the fact that configuration of the ground state is equivariant. Such an equivariant gauge configuration can be constructed using the simplest equivariant quantity $Q \equiv i\hat{x}_a\sigma_a$, where $\hat{x}_a$ is three dimensional coordinate describing S^2 and $i\sigma_a/2$ is the gauge symmetry generator. This also describes the configuration of the so called "Witten ansatz"[7].

Equivalence of pure YM theory and YMH model through the existence of extra dimensional space, has been discussed also in other models, with a little more generality, like in the coset space S/R [1] [2] model or the fuzzy version of it, $(S/R)_F$ [3, 4, 5].

We have seen a similar extra dimensions for the case of noncommutative Z_2, using the method of differential forms[8] (see also [9]) and introducing the coordinates of noncommutative Z_2 space[10]. In those papers, we have shown that difference of vortex and instanton can be considered as the difference that the space where they exist is $R^2 \times Z_2$, and R^4, and that they satisfy the same self dual equation in each space. This implies that the vortex may be treated analytically as the instanton. It would be expected that such a construction brings us many advantages to understand the solution for the vortex equation. However, the concepts of equivariance, so far, was not unambiguous in this approach.

In the present paper we would like to show that equivariance is important in this case also and we have explicitly defined Q in noncommutative Z_2 space, thus we can introduce "Witten ansatz" for the noncommutative Z_2 space, which leads to equivalence of SD equation on $R^2 \times Z_2$ to BPS equation on R^2.

Dimensional reduction through the use of equivariance principle especially for the case of S^2 and SU(2) gauge symmetry, we have to have at least a larger symmetry group that includes SU(2) as a subgroup. On the other hand, in the model that we are proposing, the extra space is a discrete Z_2, and the relevant symmetry for the equivariance is the discrete part of the gauge symmetry. Thus, there is no need to consider a larger gauge symmetry i.e. we could remain with the same gauge symmetry, and this could be the simplest possible model of this kind.

In the next sections, we recapitulate the arguments of the dimensional reduction for the YM theory on $R^2 \times S^2$ based on the equivariance principle. And based on this argument, we consider the YM theory on $R^2 \times Z_2$. The last section is devoted to discussions.

2 Equivariance on $R^2 \times S^2$ Model

As stated in the previous section, extra dimensional space orthogonal to the real space, can be left unobserved when the symmetry of the extra dimensional space were equivariant to the gauge symmetry of the real world. Instead, the gauge field on the extra dimensional space makes its appearance as the Higgs field in our world. For the invisible extra dimensional space, the gauge invariant ground state has to be an equivariant configuration. For example, let us consider SU($2N$) gauge symmetric model on $R^2 \times S^2$. In this case, S^2 can become equivariant extra dimensional space, when the gauge configuration on S^2 is described in terms of equivariant basic quantity $Q \equiv i\hat{x}_a\sigma_a$.

In order to confirm equivariance, we examine whether the following symmetry equations, which are due to Forgacs & Manton[11],

$$\begin{aligned} &\epsilon_{ijk}x_j\partial_k A_l + \epsilon_{ilk}A_k - [\mathcal{J}_i, A_l] = 0, \\ &\epsilon_{ijk}x_j\partial_k B - [\mathcal{J}_i, B] = 0 \end{aligned} \tag{2.1}$$

are satisfied or not. Here A_i is a vector and B is a scalar, $\mathcal{J}_i$ is a generator of internal space. For example, as Q is a scalar, we substitute $B = Q$ and we obtain

$$\epsilon_{ijk}x_j\partial_k Q = \epsilon_{ijk}x_j\partial_k(i\hat{x}_l\sigma_l) = i\epsilon_{ijk}x_j\sigma_k. \tag{2.2}$$

As $\mathcal{J}_i = i\sigma_i/2$, we have

$$[\mathcal{J}_i, Q] = \left[i\frac{\sigma_i}{2}, i\hat{x}_l\sigma_l\right] = i\epsilon_{ijk}\hat{x}_j\sigma_k, \tag{2.3}$$

thus Q satisfies the symmetry equation. In other words, when the ground state is described in terms of Q , the existence of S^2 could have been unobserved. Also, the gauge symmetry SU($2N$) is reduced to the smaller symmetry.

As $Q^2 = -1$, the vectors that can be constructed from Q are $\partial_a Q$ and $Q\partial_a Q$. Thus, the most general gauge configuration can be written as

$$A_a = \frac{i}{2}(\varphi_1 - 1)\partial_a Q + \frac{i}{2}\varphi_2 Q\partial_a Q. \tag{2.4}$$

Since

$$\partial_a Q = i\partial_a(\hat{x}_b\sigma_b) = \frac{i}{r}(\delta_{ab} - \hat{x}_a\hat{x}_b)\sigma_b, \tag{2.5}$$

$$Q\partial_a Q = -\frac{1}{r}(\hat{x}_c\sigma_c)(\delta_{ab} - \hat{x}_a\hat{x}_b)\sigma_b = -\frac{i}{r}\epsilon_{acd}\hat{x}_d\sigma_c, \tag{2.6}$$

we can rewrite A_a in terms of these as[12]

$$A_a = \frac{i}{2r}\left[(H^\dagger - 1)\omega_{ab} + (H - 1)\omega_{ab}^\dagger\right]\sigma_b, \quad \omega_{ab} \equiv i(\delta_{ab} - \hat{x}_a\hat{x}_b + i\epsilon_{abc}\hat{x}_c), \tag{2.7}$$

where

$$\varphi_1 = \frac{H^\dagger + H}{2}, \quad \varphi_2 = \frac{H^\dagger - H}{2} i. \tag{2.8}$$

This is the "Witten ansatz"[7]. In terms of stereographically projected coordinates

$$y = \frac{\hat{x}_1 - i\hat{x}_2}{1 - x_3} = e^{-i\varphi} \tan\frac{\theta}{2}, \quad \bar{y} = \frac{\hat{x}_1 + i\hat{x}_2}{1 - x_3} = e^{i\varphi} \tan\frac{\theta}{2}, \tag{2.9}$$

Q can be rewritten as

$$Q = \frac{i}{1 + y\bar{y}} \begin{pmatrix} -1 + y\bar{y} & 2y \\ 2\bar{y} & 1 - y\bar{y} \end{pmatrix}. \tag{2.10}$$

Using these, the Witten ansatz can be rewritten through the singular gauge transformation

$$g = \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2} \\ -e^{i\varphi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = \frac{1}{\sqrt{1 + y\bar{y}}} \begin{pmatrix} 1 & y \\ -\bar{y} & 1 \end{pmatrix}, \tag{2.11}$$

which leads to nothing but the gauge configuration of Manton & Sakai[2]

$$A_y^{\mathrm{MS}} = \frac{i}{1 + y\bar{y}} \begin{pmatrix} -\bar{y}/2 & iH^\dagger \\ 0 & \bar{y}/2 \end{pmatrix} = \frac{1}{1 + y\bar{y}} \left(-\Phi - i\Lambda\bar{y}\right), \quad \left(\Lambda \equiv \frac{1}{2}\sigma_3, \; \Phi \equiv H^\dagger \sigma_+\right) \tag{2.12}$$

$$A_{\bar{y}}^{\mathrm{MS}} = \frac{i}{1 + y\bar{y}} \begin{pmatrix} y/2 & 0 \\ -iH & -y/2 \end{pmatrix} = \frac{1}{1 + y\bar{y}} \left(\bar{\Phi} + i\Lambda y\right). \tag{2.13}$$

A generator for rotation around the third axis is

$$x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} = i\left(-y\frac{\partial}{\partial y} + \bar{y}\frac{\partial}{\partial \bar{y}}\right), \tag{2.14}$$

and a generator for the gauge transformation around σ_3 axis is $i\Lambda$. Using these, the symmetry equation[11] reads

$$\begin{aligned} \left(-y\frac{\partial}{\partial y} + \bar{y}\frac{\partial}{\partial \bar{y}}\right) A_y + \left[\frac{i}{2}\sigma_3, A_y\right] &= A_y, \\ \left(-y\frac{\partial}{\partial y} + \bar{y}\frac{\partial}{\partial \bar{y}}\right) A_{\bar{y}} + \left[\frac{i}{2}\sigma_3, A_{\bar{y}}\right] &= -A_{\bar{y}}. \end{aligned} \tag{2.15}$$

$A_y^{\mathrm{MS}}, A_{\bar{y}}^{\mathrm{MS}}$ satisfies the above equation and thus are the equivariant configurations.

Assuming that these configurations describe the ground state in S^2 it can be shown that the pure YM theory in $R^2 \times S^2$ is equivalent with the YMH theory in R^2, i.e. SD equation for this model is

$$\begin{cases} \frac{8}{(1 + y\bar{y})^2} F_{z\bar{z}} = F_{y\bar{y}}, \\ F_{z\bar{y}} = 0, \\ F_{\bar{z}y} = 0, \end{cases} \tag{2.16}$$

and substituting the above configuration, each turns into the respective BPS equation

$$\begin{cases} F_{z\bar{z}} = \dfrac{1}{8}(2i\Lambda - [\Phi, \bar{\Phi}]), \\ D_z \bar{\Phi} = 0, \\ D_{\bar{z}} \Phi = 0. \end{cases} \tag{2.17}$$

Here $z, \bar{z}$ are the coordinates on R^2, and $F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}]$, $D_z \bar{\Phi} = \partial_z \bar{\Phi} + [A_z, \bar{\Phi}]$, $D_{\bar{z}} \Phi = \partial_{\bar{z}} \Phi + [A_{\bar{z}}, \Phi]$. Namely, gauge field on S^2 is recognized as Higgs field on R^2. On the other hand, $A_z, A_{\bar{z}}$ are the gauge fields on the real space R^2, they are not transformed by Λ, i.e.

$$[\Lambda, A_z] = [\Lambda, A_{\bar{z}}] = 0. \tag{2.18}$$

Then,

$$A_z = \begin{pmatrix} A_z^L & 0 \\ 0 & A_z^R \end{pmatrix}, \quad A_{\bar{z}} = \begin{pmatrix} A_{\bar{z}}^L & 0 \\ 0 & A_{\bar{z}}^R \end{pmatrix}, \tag{2.19}$$

and $\mathrm{SU}(2N)$ gauge symmetry reduces to $\mathrm{S}\left(\mathrm{U}(N)_L \times \mathrm{U}(N)_R\right)$. As a consequence, the BPS equation is reduced further to

$$\begin{cases} F_{z\bar{z}}^L = \dfrac{1}{8}(-1 + H^\dagger H), \quad F_{z\bar{z}}^R = \dfrac{1}{8}(1 - H H^\dagger), \\ D_z H^\dagger = 0, \quad D_{\bar{z}} H = 0. \end{cases} \tag{2.20}$$

3 Equivariance on $R^2 \times Z_2$ Model

In this section we consider the case where the extra dimensional space is noncommutative Z_2. As a gauge symmetry we consider $\mathrm{SU}(N)$. In order to make the Z_2 space invisible, we choose $i\tau_3$ as an equivariant quantity Q, of which each component is $N \times N$ $\mathrm{SU}(N)$ matrix. This matrix describes the Z_2 space and is itself the block matrix each block expressing the $\mathrm{SU}(N)$.

The coordinates of noncommutative Z_2 space are described as[10]

$$w = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{\tau_1 + i\tau_2}{2}, \quad \bar{w} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{\tau_1 - i\tau_2}{2}. \tag{3.1}$$

Coordinate transformation for Z_2 space is discrete, and can be realized by $i\tau_1$, i.e.

$$(i\tau_1) w (-i\tau_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \bar{w}. \tag{3.2}$$

If we transform $Q(= i\tau_3)$ by the $i\tau_1$, we obtain

$$(i\tau_1) i\tau_3 (-i\tau_1) = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -i\tau_3, \tag{3.3}$$

which changes the sign. Then, we should choose the block components of $i\tau_3$ which are $N \times N$ matrices, so that the sign of each component can be changed by a gauge transformation, and

Q becomes equivariant. For example, when $N = 2$, we choose

$$Q = \left(\begin{array}{c|c} i\sigma_3 & 0 \\ \hline 0 & -i\sigma_3 \end{array}\right) = \left(\begin{array}{cc|cc} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ \hline 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{array}\right). \tag{3.4}$$

So, using the gauge transformation in terms of

$$g = \left(\begin{array}{c|c} i\sigma_1 & 0 \\ \hline 0 & i\sigma_1 \end{array}\right), \tag{3.5}$$

Q transformed by the spacial rotation $(i\tau_1)$ returns to the original one. That is,

$$\begin{aligned} g^\dagger(i\tau_1)Q(-i\tau_1)g &= -g^\dagger Q g \\ &= -\left(\begin{array}{c|c} (-i\sigma_1)(i\sigma_3)(i\sigma_1) & 0 \\ \hline 0 & (-i\sigma_1)(-i\sigma_3)(i\sigma_1) \end{array}\right) \\ &= -\left(\begin{array}{c|c} -i\sigma_3 & 0 \\ \hline 0 & i\sigma_3 \end{array}\right) = Q. \end{aligned} \tag{3.6}$$

Using this Q, we can construct the equivariant gauge field configuration.

As a consequence, both the Z_2 space and this discrete part of the gauge transformation become invisible, if the ground state is described by the Q.

We define the differential operators by the graded commutators[13]

$$\begin{aligned} \partial_w f &= [\bar{w}, f\} = \bar{w}f - (-1)^{[f]} f\bar{w}, \\ \partial_{\bar{w}} f &= [w, f\} = wf - (-1)^{[f]} fw. \end{aligned} \tag{3.7}$$

(about the definition, see the Appendix), where $[f]$ is $+1$, when f is even, -1 when f is odd matrix.

As $Q^2 = -1$, vectors that can be constructed from Q are $\partial_{w(\bar{w})}Q$ and $Q\partial_{w(\bar{w})}Q$. As Q is an even matrix the differential operator is a commutator, expressing in terms of τ_i we have

$$\partial_w Q = \frac{1}{2}[\tau_1 - i\tau_2, i\tau_3] = \begin{pmatrix} 0 & 0 \\ 2i & 0 \end{pmatrix}, \tag{3.8}$$

$$Q\partial_w Q = \tau_3(\tau_1 + i\tau_2) = i\tau_2 + \tau_1 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \tag{3.9}$$

similarly

$$\partial_{\bar{w}} Q = \frac{1}{2}[\tau_1 + i\tau_2, i\tau_3] = \begin{pmatrix} 0 & -2i \\ 0 & 0 \end{pmatrix}, \tag{3.10}$$

$$Q\partial_{\bar{w}} Q = -\tau_3(\tau_1 - i\tau_2) = -i\tau_2 + \tau_1 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}. \tag{3.11}$$

Thus, as the gauge configuration in Z_2 space, we have the odd matrix

$$A_w = \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix}, \ A_{\bar{w}} = \begin{pmatrix} 0 & H^\dagger \\ 0 & 0 \end{pmatrix}, \tag{3.12}$$

which are equivariant under gauge, coordinate and spin transformations like in Eq.(1). In this case, the spin transformation is given as

$$A_{w(\bar{w})} \to -A_{\bar{w}(w)}, \tag{3.13}$$

in response to the coordinate transformation, because $A_{w(\bar{w})}$ are the vector fields.

Next we define the field strength in Z_2 space. The field strength is usually defined as $F_{\mu\nu} = [D_\mu, D_\nu]$. We extend this in Z_2 space as

$$F_{XY} = [D_X, D_Y\} \equiv D_X D_Y - (-1)^{[X][Y]} D_Y D_X. \qquad (X, Y = z, \bar{z}, w, \bar{w}) \tag{3.14}$$

From the definition of graded commutator and Jacobi identity

$$[f, g\} = fg - (-1)^{[f][g]} gf, \tag{3.15}$$

$$(-1)^{[A][C]}[A, [B, C\}\} + (-1)^{[A][B]}[B, [C, A\}\} + (-1)^{[C][B]}[C, [A, B\}\} = 0, \tag{3.16}$$

we have

$$\begin{aligned} D_w D_{\bar{w}} f &= (\partial_w + A_w)(\partial_{\bar{w}} f + A_{\bar{w}} f) + A_w(\partial_{\bar{w}} f) + A_w A_{\bar{w}} f \\ &= \partial_w(\partial_{\bar{w}} f) + (\partial_w A_{\bar{w}}) f - A_{\bar{w}}(\partial_w f) + A_w(\partial_{\bar{w}} f) + A_w A_{\bar{w}} f, \end{aligned} \tag{3.17}$$

$$D_{\bar{w}} D_w f = \partial_{\bar{w}}(\partial_w f) + (\partial_{\bar{w}} A_w) f - A_w(\partial_{\bar{w}} f) + A_{\bar{w}}(\partial_w f) + A_{\bar{w}} A_w f. \tag{3.18}$$

Taking into account $\partial_{\bar{w}}\partial_w f + \partial_w \partial_{\bar{w}} f = 0$, we have

$$\begin{aligned} D_w D_{\bar{w}} f + D_{\bar{w}} D_w f &= (\partial_w A_{\bar{w}}) f + (\partial_{\bar{w}} A_w) f + A_w A_{\bar{w}} f + A_{\bar{w}} A_w f \\ &= (\partial_w A_{\bar{w}} + \partial_{\bar{w}} A_w + \{A_w, A_{\bar{w}}\}) f. \end{aligned} \tag{3.19}$$

As a result, $F_{w\bar{w}}$ can be written as $\{D_w, D_{\bar{w}}\}$.

Next, we consider $D_w D_{\bar{z}} f$ and $D_{\bar{z}} D_w f$. Taking into account that $A_{\bar{z}}$ is an even matrix, we calculate

$$\begin{aligned} D_w D_{\bar{z}} f &= (\partial_w + A_w)(\partial_{\bar{z}} f + A_{\bar{z}} f) \\ &= \partial_w(\partial_{\bar{z}} f) + (\partial_w A_{\bar{z}}) f + A_{\bar{z}} \partial_w f + A_w \partial_{\bar{z}} f + A_w A_{\bar{z}} f, \end{aligned} \tag{3.20}$$

$$\begin{aligned} D_{\bar{z}} D_w f &= (\partial_{\bar{z}} + A_{\bar{z}})(\partial_w f + A_w f) \\ &= \partial_{\bar{z}}(\partial_w f) + (\partial_{\bar{z}} A_w) f + A_w \partial_{\bar{z}} f + A_{\bar{z}} \partial_w f + A_{\bar{z}} A_w f, \end{aligned} \tag{3.21}$$

and we can consider of $[D_w, D_{\bar{z}}]$ as a field strength, i.e.

$$\begin{aligned} D_w D_{\bar{z}} f - D_{\bar{z}} D_w f &= (\partial_w A_{\bar{z}}) f - (\partial_{\bar{z}} A_w) f + A_w A_{\bar{z}} f - A_{\bar{z}} A_w f \\ &= (\partial_w A_{\bar{z}} + \partial_{\bar{z}} A_w + [A_w, A_{\bar{z}}]) f, \end{aligned} \tag{3.22}$$

thus $[D_w, D_{\bar z}] = F_{w\bar z}$. As for $F_{z\bar z}$, it is the ordinary field strength on R^2, thus $F_{z\bar z} = [D_z, D_{\bar z}]$.

From these arguments, it is appropriate to define

$$F_{XY} = [D_X, D_Y\} \equiv D_X D_Y - (-1)^{[X][Y]} D_Y D_X. \qquad (X, Y = z, \bar z, w, \bar w) \tag{3.23}$$

Now we calculate the field strength for the gauge configurations

$$A_w = \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix}, \ A_{\bar w} = \begin{pmatrix} 0 & H^\dagger \\ 0 & 0 \end{pmatrix}. \tag{3.24}$$

For example

$$\begin{aligned} F_{\bar z w} &= \partial_{\bar z} A_w - \partial_w A_{\bar z} + [A_{\bar z}, A_w] \\ &= \partial_{\bar z} \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix} - \partial_w \begin{pmatrix} A^L_{\bar z} & 0 \\ 0 & A^R_{\bar z} \end{pmatrix} + \left[\begin{pmatrix} A^L_{\bar z} & 0 \\ 0 & A^R_{\bar z} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix} \right] \\ &= \begin{pmatrix} 0 & 0 \\ \partial_{\bar z}\phi - \phi A^L_{\bar z} + A^R_{\bar z}\phi & 0 \end{pmatrix} \qquad (\phi = H + 1) \\ &= \begin{pmatrix} 0 & 0 \\ D_{\bar z}\phi & 0 \end{pmatrix}, \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} F_{w\bar w} =& \partial_w A_{\bar w} + \partial_{\bar w} A_w + \{A_w, A_{\bar w}\} \\ =& \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & H^\dagger \\ 0 & 0 \end{pmatrix} \right\} + \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix} \right\} \\ &+ \left\{ \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix}, \begin{pmatrix} 0 & H^\dagger \\ 0 & 0 \end{pmatrix} \right\} \\ =& \begin{pmatrix} \phi^\dagger\phi - 1 & 0 \\ 0 & \phi\phi^\dagger - 1 \end{pmatrix}. \end{aligned} \tag{3.26}$$

Next, in order to consider the (anti-)self dual equation we define $\widetilde{F}_{XY}$. On $R^2 \times Z_2$, since definition of F_{XY} is different from the usual one, in particular $F_{\bar w w}$ is defined as an anticommutator, we cannot write

$$\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho} F^{\lambda\rho}, \tag{3.27}$$

as in the case of R^4. Although $F_{\bar w w}$ was defined as

$$\begin{aligned} F_{\bar w w} &= \partial_{\bar w} A_w + \partial_w A_{\bar w} + \{A_{\bar w}, A_w\} \\ &= \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_w \right\} + \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_{\bar w} \right\} + \{A_{\bar w}, A_w\}, \end{aligned} \tag{3.28}$$

we can redefine this as a commutator using τ_3, as follows

$$F_{\bar w w} = \partial_{\bar w} A_w + \partial_w A_{\bar w} + \{A_{\bar w}, A_w\} \tag{3.29}$$

$$= \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_w \right] \tau_3 - \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_{\bar w} \right] \tau_3 + [A_{\bar w}, A_w]\tau_3. \tag{3.30}$$

Consequently, if we multiply this equation by τ_3 we obtain the usual expression for field strength. Taking into account that $A_z, A_{\bar{z}}$ are dual to $A_w, A_{\bar{w}}$, $\tilde{F}$ can be defined by taking dual and multiplying by τ_3 (for details, see the Appendix). Thus the (anti) self dual equations can be written as

$$F_{\bar{z}w} = 0, \quad F_{w\bar{w}} = \pm F_{z\bar{z}}\tau_3, \quad \text{etc.} \tag{3.31}$$

Substituting these equations into the ansatz, each BPS equations are equivalent to

$$D_{\bar{z}}\phi = 0, \quad F_{z\bar{z}} = \phi\phi^\dagger - 1, \quad \text{etc.} \tag{3.32}$$

$$\begin{cases} F^L_{z\bar{z}} = \phi^\dagger\phi - 1, \quad F^R_{z\bar{z}} = -\phi\phi^\dagger + 1 \\ D_z\phi^\dagger = 0, \quad D_{\bar{z}}\phi = 0 \end{cases} \tag{3.33}$$

which are consistent with [10].

4 Discussion

We have been discussing the YM theory on $R^2 \times Z_2$. In [10], by use of differential form, we have seen that the self dual equation for YM theory on $R^2 \times Z_2$ is equivalent to the BPS equation for YMH model on R^2. In the present paper, by explicitly introducing the coordinates on Z_2 space, we show the equivalence of YM theory on $R^2 \times Z_2$ and YMH model on R^2, based on the idea of equivariance. This argument is the same as in the case of showing the equivalence of YM theory on $R^2 \times S^2$ and the YMH model on R^2.

Moreover, because the Z_2 space is discrete, the required equivariance is among the discrete part of gauge symmetry and it is unnecessary to consider the larger gauge symmetry. In other words, we could remain with the same gauge symmetry and this is probably the simplest model of this kind.

The equivariant gauge configuration is described by the basic equivariant quantity Q exactly as in the case of S^2 space. As we have succeeded in introducing Q in Z_2 space, we were able to construct gauge configuration for the ground state out of Q. In other words, we succeeded in introducing the Witten ansatz in the noncommutative Z_2 space.

As stated before, we have seen in [10] that YM theory on $R^2 \times Z_2$ space is equivalent to YMH model on R^2 space. The argument was based on the differential forms, and we were able to construct the differential forms on Z_2 space using the matrices. On the other hand, in this paper, we have explicitly introduced the coordinates in Z_2 space, and consistently derived the same conclusion as in [10]. The connection of the theory based on differential forms and present theory based on the explicit coordinate is not clear, because we have not been able to construct differential forms for Z_2 space. We will discuss this point in the future publication.

Appendix

A Differential Operators on Z_2 Space

Let x_a be the coordinates in R^2, and y_α the coordinates of the curved space. The metric in the curved space can be expressed as $g_{\alpha\beta} = e^a_\alpha(y) e^a_\beta(y)$, and

$$\frac{\partial}{\partial y_\alpha} = e^a_\alpha(y) \frac{\partial}{\partial x_a}. \tag{A.1}$$

Poisson bracket is defined as

$$\{f, g\}_P \equiv \theta^{\alpha\beta}(y) \frac{\partial f}{\partial y^\alpha} \frac{\partial g}{\partial y^\beta}, \tag{A.2}$$

where

$$\theta^{\alpha\beta}(y) = \theta(y)\epsilon^{\alpha\beta}, \quad \theta(x) = \frac{1}{\sqrt{g(y)}}. \tag{A.3}$$

From these definitions, we find

$$\{y^\alpha, y^\beta\}_P = \theta^{\alpha\beta}, \quad \frac{\partial}{\partial y^\alpha} = \theta^{-1}_{\alpha\beta}\{y^\beta, \ \}_P. \tag{A.4}$$

If we consider the "correspondence principle" for the curved space, Poisson bracket can be replaced by commutator[14]

$$[w^\mu, w^\nu] = \theta^{\mu\nu}_F, \tag{A.5}$$

where w_μ's are coordinates of the fuzzified space. Then, differential operators can be written as $(\theta^{\mu\nu}_F)^{-1}[w^\nu, \ ]$. Now, regarding Z_2 space as the fuzzified space of a certain curved space, we have to find $\theta^{\mu\nu}_F$ in our case.

In our case,

$$[w, \bar{w}] = 2i[w_1, w_2] = \tau_3, \tag{A.6}$$

then, we find

$$\theta^{\mu\nu}_F = \tau_3 \epsilon^{\mu\nu}. \tag{A.7}$$

Therefore, the differential operators can be written as

$$\partial_w = -\tau_3[\bar{w}, \ ], \ \partial_{\bar{w}} = \tau_3[w, \ ], \tag{A.8}$$

and the self dual equation is

$$F_{\bar{z}z} = \tau_3 F_{\bar{w}w}. \tag{A.9}$$

This fact is interpreted as follows: in Z_2 space, the factor τ_3 is necessary for the definition of differential operators, so that the Jacobi identity and the Leibniz rule are satisfied. This

suggests that the Z_2 space is a certain curved space, and so this space possesses a measure different from that of the flat space. Therefore, τ_3 should appear in the self dual equation as a correction factor of the measure and volume element. This fact has been confirmed in the previous paper by the method using the differential fom that does not depend on the coordinates of Z_2 space.

On the other hand, these differential operators can be rewritten in terms of the graded commutators. Let A be any 2×2 matrix. A can be decomposed as

$$A = A_e + A_o, \tag{A.10}$$

where A_e is even matrix, and A_o is odd one. Namely,

$$A_e = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad A_o = \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}. \tag{A.11}$$

Then, it is easily found that the differentials of A are

$$\partial_w A = -\tau_3[\bar{w}, A] = -\tau_3[\bar{w}, A_e] - \tau_3[\bar{w}, A_o] \tag{A.12}$$

$$= \begin{pmatrix} 0 & 0 \\ a-b & 0 \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \tag{A.13}$$

$$\partial_{\bar{w}} A = \tau_3[w, A] = \tau_3[w, A_e] + \tau_3[w, A_o] \tag{A.14}$$

$$= \begin{pmatrix} 0 & b-a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}.$$

These relations can also be realized in terms of the graded commutators as

$$\partial_w A = [\bar{w}, A\} = \bar{w}A - (-1)^{[A]} A\bar{w}, \tag{A.15}$$

$$\partial_{\bar{w}} A = [w, A\} = wA - (-1)^{[A]} Aw.$$

We adopt this definition in this article.

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